

A tight lower bound for Vertex Planarization on graphs of bounded treewidth

Marcin Pilipczuk*

Abstract

In the VERTEX PLANARIZATION problem one asks to delete the minimum possible number of vertices from an input graph to obtain a planar graph. The parameterized complexity of this problem, parameterized by the solution size (the number of deleted vertices) has recently attracted significant attention. The state-of-the-art algorithm of Jansen, Lokshtanov, and Saurabh [SODA 2014] runs in time $2^{\mathcal{O}(k \log k)} \cdot n$ on n -vertex graph with a solution of size k . It remains open if one can obtain a single-exponential dependency on k in the running time bound.

One of the core technical contributions of the work of Jansen, Lokshtanov, and Saurabh is an algorithm that solves a weighted variant of VERTEX PLANARIZATION in time $2^{\mathcal{O}(w \log w)} \cdot n$ on graphs of treewidth w . In this short note we prove that the running time of this routine is tight under the Exponential Time Hypothesis, even in unweighted graphs and when parameterizing by treedepth. Consequently, it is unlikely that a potential single-exponential algorithm for VERTEX PLANARIZATION parameterized by the solution size can be obtained by merely improving upon the aforementioned bounded treewidth subroutine.

1 Introduction

In the VERTEX PLANARIZATION problem, given an undirected graph G and an integer k , our goal is to delete at most k vertices from the graph G to obtain a planar graph. If (G, k) is a YES-instance to VERTEX PLANARIZATION, then we say that G is a k -apex graph. Since many algorithms for planar graphs can be easily generalized to near-planar graphs — k -apex graphs for small values of k — this motivates us to look for efficient algorithms to recognize k -apex graphs. In other words, we would like to solve VERTEX PLANARIZATION for small values of k .

By a classical result of Lewis and Yannakakis [8], VERTEX PLANARIZATION is NP-hard when k is part of the input. Since one can check if a given graph is planar in linear time [4], VERTEX PLANARIZATION can be trivially solved in time $\mathcal{O}(n^{k+1})$, where $n = |V(G)|$, that is, in polynomial time for every fixed value of k . However, such an algorithm is impractical even for small values of k ; a question for a faster algorithm brings us to the realms of *parameterized complexity*.

In the parameterized complexity, every problem comes with a *parameter*, being an additional complexity measure of input instances. The central notion is a *fixed-parameter algorithm*: an algorithm that solves an instance x with parameter k in time $f(k)|x|^{\mathcal{O}(1)}$ for some computable function f . Such a running time bound, while still super-polynomial (the function f is usually exponential), is considered significantly better than say $\mathcal{O}(|x|^k)$, as it promises much faster algorithms for moderate values of k and large instances. We refer to recent textbooks [1, 2] for a more broad introduction to parameterized complexity.

Due to the aforementioned motivation, it is natural to consider the solution size k as a parameter for VERTEX PLANARIZATION, and ask for a fixed-parameter algorithm. Since, for a fixed value of k , the class of all k -apex graphs is closed under taking minors, the graph minor theory of Robertson and Seymour immediately yields a fixed-parameter algorithm, but with enormous dependency on the parameter in the running time bound.¹ The quest for an explicit and faster fixed-parameter algorithm for VERTEX PLANARIZATION has attracted significant attention in the parameterized complexity community

*Institute of Informatics, University of Warsaw, Poland, malcin@mimuw.edu.pl

¹Formally, this algorithm is non-uniform, that is, it requires an external advice depending on the parameter only. However, we can obtain a uniform algorithm using the techniques of Fellows and Langston [3].

in the recent years. First, Marx and Schlotter [12] obtained a relatively simple algorithm, with doubly-exponential dependency on the parameter and n^2 dependency on the input size in the running time bound. Later, Kawarabayashi [7] obtained a fixed-parameter algorithm with improved linear dependency on the input size, at the cost of worse dependency on the parameter. Finally, Jansen, Lokshtanov, and Saurabh [6] developed an algorithm with running time bound $2^{\mathcal{O}(k \log k)} \cdot n$, improving upon all previous results.

As noted in [6], a simple reduction shows that VERTEX PLANARIZATION cannot be solved in time $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ unless the Exponential Time Hypothesis fails. Informally speaking, the Exponential Time Hypothesis (ETH) [5] asserts that the satisfiability of 3-CNF formulae cannot be verified in time subexponential in the number of variables. In the recent years, a number of tight bounds for fixed-parameter algorithms have been obtained using ETH or the closely related Strong ETH; we refer to [9, 11] for an overview. In this light, it is natural to ask for tight bounds for fixed-parameter algorithms for VERTEX PLANARIZATION. In particular, [6] asks for a single-exponential (i.e., with running time bound $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$) algorithm.

The core subroutine of the algorithm of Jansen, Lokshtanov, and Saurabh, is an algorithm that solves VERTEX PLANARIZATION in time $2^{\mathcal{O}(w \log w)} \cdot n$ on graphs of treewidth w . A direct way to obtain a single-exponential algorithm for VERTEX PLANARIZATION parameterized by the solution size would be to improve the running time of this bounded treewidth subroutine to $2^{\mathcal{O}(w)} \cdot n^{\mathcal{O}(1)}$. In this short note we show that such an improvement is unlikely, as it would violate the Exponential Time Hypothesis.

Theorem 1. *Unless the Exponential Time Hypothesis fails, there does not exist an algorithm that solves VERTEX PLANARIZATION on n -vertex graphs of treewidth at most w in time $2^{o(w \log w)} n^{\mathcal{O}(1)}$.*

In fact, our lower bound holds even for a more restrictive parameter of *treedepth*, instead of treewidth.

While Theorem 1 does not exclude the possibility of a $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ -time algorithm for VERTEX PLANARIZATION, it shows that to obtain such a running time one needs to circumvent the usage of bounded-treewidth subroutine on graphs of treewidth $\Omega(k)$ in the algorithm of Jansen, Lokshtanov, and Saurabh.

The remainder of this paper is devoted to the proof of Theorem 1.

2 Lower bound

We base our reduction on the framework for proving superexponential lower bounds introduced by Lokshtanov, Marx, and Saurabh [10]. For an integer k , by $[k]$ we denote the set $\{1, 2, \dots, k\}$. Consequently, $[k] \times [k]$ is a $k \times k$ table of elements with *rows* being subsets of the form $\{i\} \times [k]$, and *columns* being subsets of the form $[k] \times \{i\}$. We start from the following auxiliary problem.

$k \times k$ PERMUTATION CLIQUE	Parameter: k
Input: An integer k and a graph G with vertex set $[k] \times [k]$.	
Question: Is there a k -clique in G with exactly one element from each row and exactly one element from each column?	

As proven in [10], an $2^{o(k \log k)}$ -time algorithm for $k \times k$ PERMUTATION CLIQUE would violate ETH. Hence, to prove Theorem 1, it suffices to prove the following.

Lemma 2. *There exists a polynomial time algorithm that, given an instance (G, k) of $k \times k$ PERMUTATION CLIQUE, outputs an equivalent instance (H, ℓ) of VERTEX PLANARIZATION where the treedepth of the graph H is bounded by $\mathcal{O}(k)$.*

That is, as announced in the introduction, we in fact prove a stronger variant of Theorem 1, refuting an existence of a $2^{o(w \log w)} n^{\mathcal{O}(1)}$ -time algorithm for VERTEX PLANARIZATION parameterized by the treedepth of the input graph. Recall that the treedepth of a graph G , denoted $\text{td}(G)$, is always not smaller than the treewidth of G , and satisfies the following recursive formula.

Lemma 3 ([13]). *The treedepth of an empty graph is 0, and the treedepth of a one-vertex graph equals 1. The treedepth of a disconnected graph G equals the maximum of the treedepth of the connected components of G . The treedepth of a connected graph G is equal to*

$$\text{td}(G) = 1 + \min_{v \in V(G)} \text{td}(G - \{v\}).$$

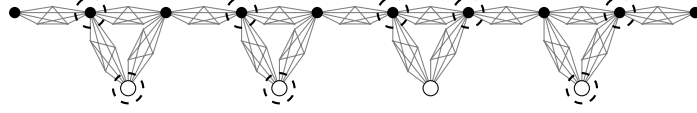


Figure 1: Choice gadget C_4 . The vertices a_i are black and the vertices b_j are white. A minimum solution that leaves b_3 undeleted is marked with dashed circles.

We refer to the textbook [14] for more information on treedepth.
The rest of this section is devoted to the proof of Lemma 2.

2.1 One-in-many gadget

We begin with a description of a gadget that allows us to encode a choice among many options.

Given two vertices x and y , by *introducing a K_5 -edge xy* we mean the following operation: we introduce three new vertices z_1, z_2 and z_3 and make x, y, z_1, z_2, z_3 a clique. Note that in every solution to VERTEX PLANARIZATION, at least one of the vertices of the set $\{x, y, z_1, z_2, z_3\}$ needs to be deleted. As we do not add any more edges incident to any vertex z_i , $i = 1, 2, 3$, we may safely restrict ourselves to solutions to VERTEX PLANARIZATION that contain x or y and do not contain any of the vertices z_i , $i = 1, 2, 3$. That is, we treat the vertices z_i as undeletable vertices, and henceforth by a “solution to VERTEX PLANARIZATION” we mean a solution not containing any such vertex.

For an integer $s \geq 1$, we define an s -choice gadget C_s as follows. We start with $3s+2$ vertices denoted a_i for $0 \leq i \leq 2s+1$ and b_j for $1 \leq j \leq s$. Then, for each $0 \leq i < 2s+1$ we introduce a K_5 -edge $a_i a_{i+1}$ and for each $1 \leq j < s$ we introduce two K_5 -edges $b_j a_{2j-1}$ and $b_j a_{2j}$. Any choice gadget created in the construction will be attached to the rest of the graph using the vertices b_j ; informally speaking, in any optimal solution, exactly one vertex b_j remains undeleted. We summarize the properties the s -choice gadget in the following lemma; see Figure 1 for an illustration.

Lemma 4. *For an s -choice gadget C_s , the following holds.*

1. *A minimum solution to VERTEX PLANARIZATION on C_s consists of $2s$ vertices.*
2. *For every $1 \leq j \leq s$ there exists a minimum solution X to VERTEX PLANARIZATION on C_s that contains all vertices $b_{j'}$ for $j' \neq j$.*
3. *In every minimum solution to VERTEX PLANARIZATION on C_s , at least one vertex b_j remains undeleted.*
4. *The treedepth of the s -choice gadget is $\mathcal{O}(\log s)$. Furthermore, the same treedepth bound holds for a graph constructed from an s -choice gadget by, for every $1 \leq j \leq s$, introducing a constant-size graph G_j and identifying one vertex of G_j with the vertex b_j .*

Proof. First, note that for every $1 \leq j \leq s$, the set

$$\{b_{j'} : j' \neq j\} \cup \{a_{2j'-1} : 1 \leq j' \leq j\} \cup \{a_{2j'} : j \leq j' \leq s\}$$

is a solution to VERTEX PLANARIZATION on C_s of size $2s$ that contains all vertices $b_{j'}$ except for b_j . Moreover, observe that, due to K_5 -edges, for every $1 \leq j \leq s$, any solution to VERTEX PLANARIZATION on C_s needs to delete at least two vertices from the set $\{a_{2j-1}, a_{2j}, b_j\}$, and consequently has size at least $2s$. This settles the first two claims.

For the third claim, note that any vertex cover of a path of length $2s+1$ needs to contain at least $s+1$ vertices, and consequently a solution to VERTEX PLANARIZATION on $C_s \setminus \{b_j : 1 \leq j \leq s\}$ needs to contain at least $s+1$ vertices. Thus, any solution to VERTEX PLANARIZATION on C_s that contains $\{b_j : 1 \leq j \leq s\}$ contains at least $2s+1$ vertices. This settles the third claim.

We prove the last claim by induction on s . For $s = \mathcal{O}(1)$, the gadget and the attached graphs G_j are of constant size, and the treedepth is constant. Otherwise, for an s -choice gadget C_s , we delete the three vertices z_1, z_2, z_3 from the K_5 -edge between $a_{2\lfloor s/2 \rfloor}$ and $a_{2\lfloor s/2 \rfloor + 1}$. Note that the gadget splits into two connected components, both being subgraphs of a graph in question constructed from a $\lceil s/2 \rceil$ -choice gadget. The treedepth bound $\mathcal{O}(\log s)$ follows from Lemma 3. \square

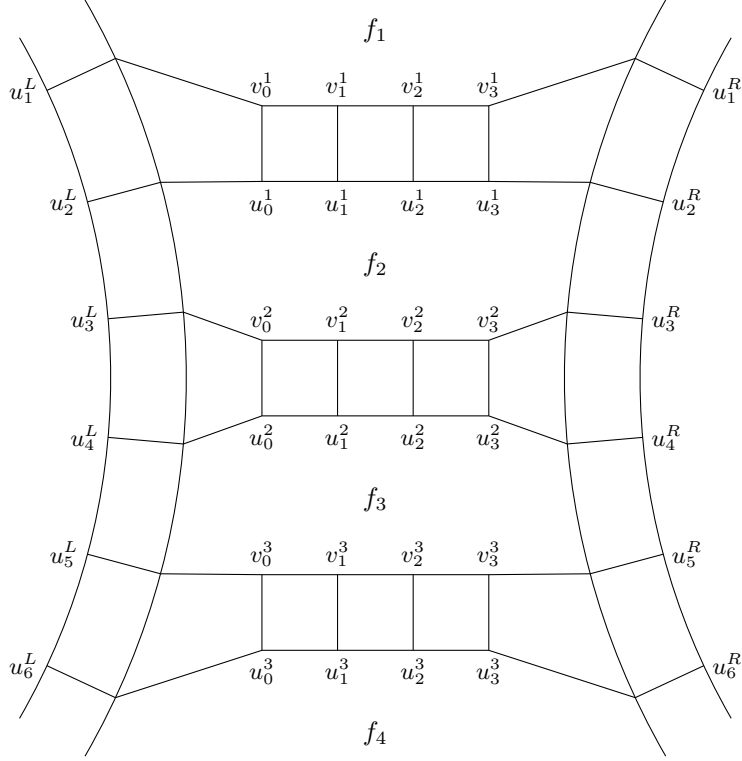


Figure 2: Part of the frame graph H_F with its unique embedding.

2.2 Construction

We now give a construction of the VERTEX PLANARIZATION instance (H, ℓ) , given a $k \times k$ PERMUTATION CLIQUE instance (G, k) . Let $m = |E(G)|$ and assume $k \geq 2$.

First, we introduce a *frame graph* H_F . A *ladder of length n* is a $2n$ -vertex graph that consists of two paths v_1, v_2, \dots, v_n and u_1, u_2, \dots, u_n together with edges $v_i u_i$ for $1 \leq i \leq n$. A *cycle ladder of length n* additionally contains edges $v_n v_1$ and $u_n u_1$, that is, v_1, v_2, \dots, v_n and u_1, u_2, \dots, u_n are in fact cycles. The frame graph H_F consists of two cycle ladders of length $2k$, with vertex sets $\{v_i^\Gamma, u_i^\Gamma : 1 \leq i \leq 2k\}$ for $\Gamma \in \{L, R\}$, and of k ladders of length 4 with vertex sets $\{v_i^\alpha, u_i^\alpha : 0 \leq i \leq 3\}$ for $1 \leq \alpha \leq k$, connected with edges $v_{2\alpha-1}^L v_0^\alpha, v_{2\alpha}^L u_0^\alpha, v_{2\alpha-1}^R v_3^\alpha, v_{2\alpha}^R u_3^\alpha$ for each $1 \leq \alpha \leq k$ (see Figure 2). Note that H_F is 3-edge-connected and hence has a unique planar embedding. By f^α we denote the face of the embedding of H_F that is incident to all vertices $v_i^\alpha, 0 \leq i \leq 3$.

Second, we introduce k vertices $x^\beta, 1 \leq \beta \leq k$. Our intention is to ensure that in any solution to VERTEX PLANARIZATION on (H, ℓ) , no vertex x^β nor no vertex from the frame H_F will be deleted, and each vertex x^β will be embedded into a different face f^α . The choice of which vertex x^β is embedded into which face will correspond to a choice of the vertices of the clique in the instance (G, k) of $k \times k$ PERMUTATION CLIQUE. We now force such a behavior with some gadgets.

For each $i = 1, 2, 3$, perform the following construction. For every $1 \leq \alpha, \beta \leq k$, introduce a vertex $y_i^{\alpha, \beta}$ incident to v_i^α and x^β . Moreover, for every $1 \leq \beta \leq k$, introduce a k -choice gadget $C^{i, \beta}$ and for each $1 \leq \alpha \leq k$ identify the vertex b_α of $C^{i, \beta}$ with $y_i^{\alpha, \beta}$. See Figure 3. Informally speaking, by the properties of the k -choice gadget, each vertex x^β needs to select one face f^α that will contain it in the planar embedding. The fact that the construction is performed three times ensures that no face f^α is chosen by two vertices x^β , as otherwise a $K_{3,3}$ -minor will be left in the graph.

Let us now move to the description of the encoding of the edges of G . For an edge $e \in E(G)$, let $e = (p(e), \gamma(e))(q(e), \delta(e))$ where $p(e) < q(e)$ (note that edges e with $p(e) = q(e)$ are irrelevant to the problem, and hence we may assume there are no such edges). For $1 \leq p < q \leq k$, we define $E(p, q) = \{e \in E(G) : (p(e), q(e)) = (p, q)\}$. For each edge $e \in E(G)$, we introduce in H three vertices

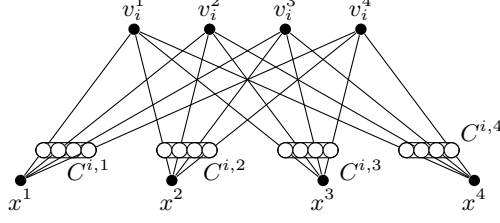


Figure 3: Connections between the frame and vertices x^β that ensure that every vertex x^β is embedded into a different face f^α . Vertices $y_i^{\alpha,\beta}$ are depicted white.

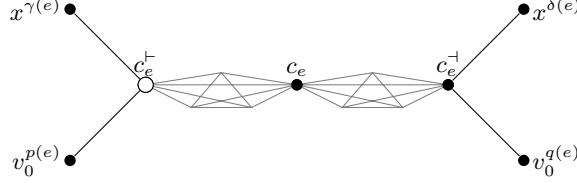


Figure 4: Gadget introduced for an edge $e \in E(G)$. The vertex c_e^+ , marked white, is part of the choice gadget $\hat{C}^{p(e),q(e)}$.

c_e^+, c_e, c_e^- , four edges $v_0^{p(e)} c_e^+, x^{\gamma(e)} c_e^+, v_0^{q(e)} c_e^-, x^{\delta(e)} c_e^-$ and two K_5 -edges $c_e^+ c_e$ and $c_e^- c_e$ (see Figure 4). Moreover, for every $1 \leq p < q \leq k$, introduce a $|E(p,q)|$ -choice gadget $\hat{C}^{p,q}$ and for every $e \in E(p,q)$ identify c_e^+ with a distinct vertex b_j of $\hat{C}^{p,q}$. Informally speaking, for each edge e we need either to delete c_e^+ and c_e^- or only c_e ; however, the second option is only possible if $x^{\gamma(e)}$ is embedded into $f_{p(e)}$ and at the same time $x^{\delta(e)}$ is embedded into $f_{q(e)}$. The choice gadget $\hat{C}^{p,q}$ ensures that we can choose the second, cheaper option only once per each pair (p,q) .

We set $\ell = 3m + 6k^2$. This completes the description of the instance (H, ℓ) . Note that the budget ℓ is tight: it allows only to choose a minimum solution in all introduced choice gadgets, and one endpoint of each K_5 -edge $c_e^\pm c_e$.

2.3 Treedepth bound

Lemma 5. *The treedepth of H is $\mathcal{O}(k)$.*

Proof. We use the recursive formula of Lemma 3. First, we delete from H all vertices of the frame H_F and all vertices x^β . Note that we have deleted only $17k$ vertices in this manner. By Lemma 3, it suffices to show that every connected component of the remaining graph has treedepth $\mathcal{O}(k)$; we will in fact show a stronger bound of $\mathcal{O}(\log k)$.

Observe that the remaining graph contains two types of connected components. The first type are the k -choice gadgets $C^{i,\beta}$ for $1 \leq i \leq 3$ and $1 \leq \beta \leq k$; by Lemma 4, every such gadget has treedepth $\mathcal{O}(\log k)$. The second type are the $|E(p,q)|$ -choice gadgets $\hat{C}^{p,q}$ for $1 \leq p < q \leq k$, together with the vertices c_e^+, c_e, c_e^- and the K_5 -edges between them. As $|E(p,q)| \leq k^2$ for every $1 \leq p < q \leq k$, by Lemma 4 the treedepth of these connected components is also $\mathcal{O}(\log k)$. This finishes the proof of the lemma. \square

2.4 Equivalence

In the following two lemmata we show the equivalence of the constructed VERTEX PLANARIZATION instance (H, ℓ) and the input $k \times k$ PERMUTATION CLIQUE instance (G, k) , completing the proof of Lemma 2 and of Theorem 1.

Lemma 6 (Completeness). *If (G, k) is a YES-instance to $k \times k$ PERMUTATION CLIQUE, then (H, ℓ) is a YES-instance to VERTEX PLANARIZATION.*

Proof. Let $\rho : [k] \rightarrow [k]$ be a solution to $k \times k$ PERMUTATION CLIQUE on (G, k) , that is, ρ is a permutation of $[k]$ and $K := \{(p, \rho(p)) : 1 \leq p \leq k\}$ is a clique in G . Consider the following set $X \subseteq V(H)$.

1. For each $i = 1, 2, 3$ and $1 \leq \alpha \leq k$, X contains a minimum solution (i.e., of size $2k$) to VERTEX PLANARIZATION in the gadget $C^{i, \rho(\alpha)}$ that contains all vertices $y_i^{\alpha', \rho(\alpha)}$ for $1 \leq \alpha' \leq k$ except for $y_i^{\alpha, \rho(\alpha)}$.
2. For each $1 \leq p < q \leq k$, denote by $e(p, q)$ the unique edge in $E(p, q)$ such that $e(p, q) = (p, \rho(p))(q, \rho(q))$. Then X contains a minimum solution to VERTEX PLANARIZATION in the gadget $\hat{C}^{p, q}$ that contains all vertices c_e^\pm for $e \in E(p, q)$ except for $c_{e(p, q)}^\pm$, the vertex $c_{e(p, q)}$ and all vertices c_e^\pm for $e \in E(p, q)$ except for $c_{e(p, q)}^\pm$.

Note that we have introduced $3 \cdot k \cdot 2k = 6k^2$ vertices in the first step and $3m$ vertices in the second step. Hence, $|X| = 3m + 6k^2 = \ell$. We now argue that $H \setminus X$ is planar. It suffices to prove it for each 2-connected component of $H \setminus X$. Note that the claim is trivial or follows from Lemma 4 for each 2-connected component of $H \setminus X$ except for the one that contains the frame H_F .

Consider the unique planar embedding of H_F and embed into each face f^α the vertex $x^{\rho(\alpha)}$. Note that each vertex x^β is embedded into a different face. It is straightforward to verify that $x^{\rho(\alpha)}$ can be embedded into f^α together with vertices $y_i^{\alpha, \rho(\alpha)}$ for $i = 1, 2, 3$ and the vertices c_e^\pm or c_e^\mp that correspond to the edges of $G[K]$ incident to $(\alpha, \rho(\alpha))$. Note that all other vertices of $H \setminus X$ lie in different 2-connected components than H_F and, consequently, $H \setminus X$ is planar. \square

Lemma 7 (Soundness). *If (H, ℓ) is a YES-instance to VERTEX PLANARIZATION, then (G, k) is a YES-instance to $k \times k$ PERMUTATION CLIQUE.*

Proof. Let $X \subseteq V(H)$ be such that $|X| \leq \ell$ and $H \setminus X$ is planar. By Lemma 4, X needs to contain at least $2k$ vertices from each gadget $C^{i, \beta}$ for $i = 1, 2, 3$ and $1 \leq \beta \leq k$; note that there are $3k$ such gadgets. Moreover, X needs to contain at least $2|E(p, q)|$ vertices from each gadget $\hat{C}^{p, q}$ for $1 \leq p < q \leq k$, which totals to at least $2m$ vertices. Finally, for each $1 \leq i \leq m$, X needs to contain at least one vertex of the K_5 -edge $c_e c_e^\perp$. As $|X| \leq \ell = 6k^2 + 3m$, we infer that $|X| = \ell$ and X contains a minimum solution to VERTEX PLANARIZATION on each introduced choice gadget, and exactly one vertex from the pair $\{c_e, c_e^\perp\}$ for each $1 \leq i \leq m$. In particular, X does not contain any vertex of the frame graph H_F , nor any vertex x^β , $1 \leq \beta \leq k$.

By the properties of the choice gadget $C^{i, \beta}$, there exists $\alpha(i, \beta)$ such that $y_i^{\alpha(i, \beta), \beta} \notin X$. Recall that X does not contain any vertex of H_F , and H_F is 3-edge-connected and, hence, admits a unique planar embedding depicted on Figure 2. As X does not contain x^β , we infer that $\alpha(i, \beta) = \alpha(i', \beta)$ for every $i, i' \in \{1, 2, 3\}$. Hence, we may suppress the argument i and henceforth analyze function $\alpha(\beta)$ such that $y_i^{\alpha(\beta), \beta} \notin X$ for every $1 \leq \beta \leq k$ and $i = 1, 2, 3$. Note that x^β needs to be embedded into the face $f^{\alpha(\beta)}$ of H_F in any planar embedding of $H \setminus X$.

We now argue that $\alpha(\cdot)$ is a permutation. By contradiction, let $\alpha(\beta) = \alpha(\beta')$ for some $\beta \neq \beta'$. It is straightforward to verify that the following sets form a model of a $K_{3,3}$ minor in $H \setminus X$, contradicting its planarity.

1. $\{v_i^\alpha, y_i^{\alpha, \beta}, y_i^{\alpha, \beta'}\}$ for $i = 1, 2, 3$;
2. $\{x^\beta\}$ and $\{x^{\beta'}\}$;
3. $\{u_1^\alpha, u_2^\alpha, u_3^\alpha\}$.

We infer that $\alpha(\cdot)$ is a permutation of $[k]$. We claim $K := \{(\alpha(\beta), \beta) : 1 \leq \beta \leq k\}$ induces a clique in G . That is, that $\rho := \alpha^{-1}$ is a solution to $k \times k$ PERMUTATION CLIQUE on (G, k) .

Pick arbitrary $1 \leq p < q \leq k$. Our goal is to prove that $(p, \rho(p))(q, \rho(q)) \in E(G)$. Consider the choice gadget $\hat{C}^{p, q}$. By Lemma 4, there exists $e \in E(p, q)$ such that $c_e^\perp \notin X$. Consequently, $c_e \in X$ due to the K_5 -edge $c_e^\perp c_e$, and thus $c_e^\perp \notin X$. As c_e^\perp is adjacent to both v_0^p and $x^{\gamma(e)}$, we infer that $x^{\gamma(e)}$ is embedded into the face f^p and, consequently, $\alpha(\gamma(e)) = p$. Symmetrically, we infer that $\alpha(\delta(e)) = q$. Thus, $e = (p, \rho(p))(q, \rho(q)) \in E(G)$ and the lemma is proven. \square

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References

- [1] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
- [2] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.
- [3] M. R. Fellows and M. A. Langston. On search, decision, and the efficiency of polynomial-time algorithms. *J. Comput. Syst. Sci.*, 49(3):769–779, 1994.
- [4] J. E. Hopcroft and R. E. Tarjan. Efficient planarity testing. *J. ACM*, 21(4):549–568, 1974.
- [5] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, 2001.
- [6] B. M. P. Jansen, D. Lokshtanov, and S. Saurabh. A near-optimal planarization algorithm. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 1802–1811. SIAM, 2014.
- [7] K. Kawarabayashi. Planarity allowing few error vertices in linear time. In *50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA*, pages 639–648. IEEE Computer Society, 2009.
- [8] J. M. Lewis and M. Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *J. Comput. Syst. Sci.*, 20(2):219–230, 1980.
- [9] D. Lokshtanov, D. Marx, and S. Saurabh. Lower bounds based on the exponential time hypothesis. *Bulletin of the EATCS*, 105:41–72, 2011.
- [10] D. Lokshtanov, D. Marx, and S. Saurabh. Slightly superexponential parameterized problems. In D. Randall, editor, *SODA*, pages 760–776. SIAM, 2011.
- [11] D. Marx. What’s next? Future directions in parameterized complexity. In H. L. Bodlaender, R. Downey, F. V. Fomin, and D. Marx, editors, *The Multivariate Algorithmic Revolution and Beyond - Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday*, volume 7370 of *Lecture Notes in Computer Science*, pages 469–496. Springer, 2012.
- [12] D. Marx and I. Schlotter. Obtaining a planar graph by vertex deletion. *Algorithmica*, 62(3-4):807–822, 2012.
- [13] J. Nešetřil and P. O. de Mendez. Tree-depth, subgraph coloring and homomorphism bounds. *Eur. J. Comb.*, 27(6):1022–1041, 2006.
- [14] J. Nešetřil and P. O. de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012.